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Cognitive Analysis of Constructing Algebraic Proof Processes: A Mixed Method Research *

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Abstract

This study aimed in order to investigate cognitive aspects of the processes of constructing algebraic proofs of pre-service and service secondary math teachers. We used explanatory sequential design from mixed research methods in the study. Secondary math teachers working in the province east of Turkey and pre-service secondary math teacher in this province participated to the study. Quantitative data of the study were collected through a "Proving Diagnostic Test" and qualitative data were gathered from the participants through a think aloud protocol. We used two different activity cards that each of them included one algebraic proof question in the think aloud protocol. Quantitative data were analyzed using descriptive and inferential statistics. Content analysis was applied to qualitative data. The study show that the cognitive skills performed by pre-service and service secondary math teachers emerge five categories: read the proposition of the proof, evaluating the correctness, determining strategies, carry out plans and heuristic shortcuts thinking strategy. Eight categories were identified in the theme (context of) of meta-cognitive skills. These categories are: facilitating the operations, questioning, awareness, planning, strategy determination, controlling, correlating, and analogical reasoning.

Keywords

Cognition
Metacognition
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Introduction

Proofs are important concepts for mathematics education. Because the proof will facilitate the understanding of mathematical concepts that will provide the rationale for the mathematical expression, formula or theorems (Berggren, 1990; Solow, 2014). Furthermore, the proof will contribute to the development of mathematical reasoning and analytical thinking skills by positively affecting the mental processes of individuals (National Council of Teachers of Mathematics [NCTM], 2000, s. 56; Rice, 2014). Considering that the proof is a process requiring advanced knowledge and high level cognitive skills (Duval, 1999), it is understood that it is not easy to understand the proof for students and teachers (Aljaberi, 2014). For this reason, in order to prove, individuals must employ their cognitive and metacognitive skills after fill the deficiency of theoretical knowledge (Zazkis, Weber, & Mejía-Ramos, 2016). Because some facts that are important for the process of proving require cognitive and metacognitive skills -determining why a proposal is correct or not, or whether it can be generalised, knowing why it does it- (Özsoy, 2008). In this context, investigating of the process of proving in terms of cognitive perspective reveals the processes of thinking that individuals use to prove. Moreover, this can help to learn how to prove. Indeed, Selden and Selden (2015) pointed out firstly can be detected cognitive factors in proving for learning and teaching of proof. Examining the proving process by cognitive will help to create a theoretical framework for the teaching of proof. Because revealing the process from the reading of the propositions of individuals to the evaluation of the proof, they will provide an understanding of the processes of proof by revealing what skills they are working in this process.

Mathematical Proof

Proof in mathematics education convinces the students of the correctness or inaccuracy of the propositions which have been shown before (Aydoğdu-İskenderoğlu, 2016). The process of constructed mathematical proof is called proving. Yıldırım (2000) defines proving as "a judgment, an assertion, or a consequence to impose the correctness (or falsehood) with sufficient evidence." Algebraic proof, mathematical validity of the necessary logical inferences and using symbolic expressions (mathematical notations) can be explained as proof (Arslan & Yıldız, 2010). In mathematics, the way in which mathematical proofs are used differs because everyone has different knowledge, reasoning and thought. The different ways that individuals use in the process of proving are generally referred to as proof schemes (Sowder & Harel, 1998). Harel and Sowder (1998) categorized the proof schemes as "External", "Experimental " and "Analytical ". Under these schemes, rituals have determined that students evaluate the correctness of the proof according to the schemes of authoritarian, symbolic, inductive, perceptual, transformational and axiomatic proofs. In addition to saying that a proof is true, it is necessary to use advanced cognitive skills (Fitzgerald, 1996; Senk, 1985) because it makes it necessary to explain why it is right (Almeida, 2000); reasoning, inference, in-depth understanding, and comprehension of mathematical relations (Berggren, 1990; Solow, 2014).

Proof and Mathematical Thinking

The mental representation process requires thinking skills, such as problem-solving, abstraction, and reasoning (Solso, Maclin, & Maclin, 2014; Yıldırım, 2000). Some the thinking skills include heuristic shortcuts, perceptual set-ups, analogical reasoning, creative thinking, inductive reasoning, and deductive reasoning. Heuristic shortcuts are defined as operations that reduce the number of transactions to provide a faster, easier solution to the problem, and these shortcuts are often used in decision making and developing conclusions (Plotnik, 2009). However, they do not guarantee an end result, and sometimes, they move away from it (Bruning, Schraw, & Norby, 2014). A perceptual set-up is a type of cognitive activity that occurs while individuals think and perceive in the problem-solving process (Solso et al., 2014). Analytical reasoning is the adaptation of a solution path that results in the solution to a new problem (Plotnik, 2009). Therefore, the solution methods that have similarities in several features has not been known by using the similarity of the solutions and reaching to the unknown (Fersahoğlu, 2015). Creative thinking reorganizes information and produces innovative ideas and unusual solutions (Sternberg, 2000). Goldstein (2013) argued that creative thinking is usually the result of analogical reasoning. Creative thinking is divided into divergent and convergent thinking (Woolfolk-Hoy, 2015). Divergent thinking produces a number of solutions or ideas (Plotnik, 2009). Convergent thinking reduces several solutions to a single solution, while divergent thinking begins at one point and increases the number of solution paths (Woolfolk-Hoy, 2015). Deductive reasoning is based on the reduction of the general situation (Fersahoglu, 2015). Deductive reasoning has two premise propositions and deduction proposals (Goldstein, 2013). In induction, the individual uses one or more specific ideas or solutions to propose a conclusion (Goldstein, 2013). Inductive reasoning requires inference and takes advantage of an individual's experience (Solso et al., 2014). It is not the validity, as in deductive reasoning, in inductive reasoning that matters, but the weakness or strength of the argument (Goldstein, 2013).

Cognition and Metacognition

As explained above, perhaps the most important of the components of mathematical thinking is the metacognition. Metacognition is usually a concept that is evaluated together with cognition and difficult to distinguish from cognition. Because cognition and metacognition are two very close concepts. Cognition is information and activities that the individual uses in performing certain tasks-including mental processes (Özdemir & Sarı, 2016). Forrest-Pressley and Gillies (1983, p. 134) cited cognition as the skills and strategies used by the reader. Metacognition defines the individual as aware of the learning process and regulates their cognition (Akin, 2013). In other words, metacognition is the task control center of the cognitive system (Bruning et al., 2014). Metacognition is a part of a person's ability to perform any task (problem solving, reading, proofing) or organizing their own learning (Paris & Winograd, 1998). For example, a teacher who knows that students can not actually hold their names shows that students have the upper hand to write their names on the paper and ask them to hang them on their hands (Bruning et al., 2014). Metacognition is part of the person's ability to perform any task (such as problem solving, reading, proofing) or regulated his/her own learning (Paris & Winograd, 1998). For example, a teacher who knows that the students cannot remember their names can write their names on a paper and ask them to hang the paper on the students' collar. This shows that the teacher has metacognitive skills (Bruning et al., 2014).

The distinction between cognition and metacognition is related to how information is used (Özdemir & Sarı, 2016) what is the object of the process (Karakelle & Saraç, 2010). The skills required to complete a task (knowing the strategies, using the representations) are cognitive; the awareness of these skills is metacognitive (Okçu & Kahyaoglu, 2007; Özdemir & Sarı, 2016). Weinert (1987) explains cognition as a quadratic cognition, i.e. thinking about thinking, to cognition and metacognition. When working on a topic or performing a task, notes taken, cognitive processing mistakes are made, and the result reached by the target is metacognitive. When performing the same task, instead of taking notes, copying, using memorized ready-to-use formulas, etc., are cognitive. These skills provide a way to distinguish cognitive and metacognitive skills. This distinction can explained as follows; cognitive skills use to perform tasks without having to use any similar skills, while metacognitive skills require awareness or critical thinking (Akin, 2013).

Raising individuals who can think correctly (using mathematical thinking skills) using high level cognitive skills is a difficult condition for mathematics educators (Hamilton, Kelly, & Sloane, 2002). First of all, professional development of mathematics teachers should be sufficient (Rosenholtz, 1985). Various variables such as the duration of vocational experience of mathematics teachers and the trainings they receive are at the forefront in this assessment (Copur-Gencturk & Lubienski, 2013). These variables can be examined through developmental studies. Comparing the pre-service teacher according to the first and last grade levels (Imamoglu & Yontar-Toğrol, 2010), with the knowledge they have acquired in secondary education, by adding the knowledge they have received in the universities to the concept of proof. In this study, first and last grade pre-service teachers were included in the study in order to examine the differentiation status of proving skills according to the traineeship of the pre-service teachers. We think that first grade pre-service teachers can prove their knowledge by the information they bring from secondary education. Because these teacher candidates have completed secondary education and are at the beginning of higher education. Last grade teacher candidates completed the required field courses in higher education. For this reason, it is expected that teacher information in secondary education is added to the knowledge they have obtained in higher education. In addition, since teachers do not use most of the courses they have taken in their undergraduate studies and the secondary curriculum does not adequately cover the notion of proof (Knuth, 2002), it can be observed that teachers forget their old knowledge. This can cause the individual to prove himself by using mathematical reasoning, using metacognitive knowledge and skill, without formal proof. Because teachers with sufficient professional development can improve their students' knowledge and skills in a positive way. Mathematics knowledge must be taken into account when assessing the professional development of mathematics teachers. For this reason, teachers and pre-service teachers were included in the study and tried to examine how the situation changed in a developmental way.

Literature Review

The literature show that studies usually involve only quantitative or qualitative methods. Quantitative studies are generally studied in terms of scale development, descriptive research, and relational research (Komatsu, 2016; Yang & Lin, 2008; Yang, 2012). Qualitative researches are conducted to examine opinions or examine the knowledge and skills of the prosecution process (Alcock, 2010; Alcock & Weber, 2005; Almeida, 2000; Ceylan, 2012; Doruk & Kaplan, 2015; Fukawa-Connelly, 2012; Knuth, 2002 ; Lesseig, 2016; Martin & Harel, 1989; Stylianides & Stylianides, 2009). In this study, mixed research method was used by using qualitative and quantitative methods. This will allow analytic generalization from a wider sample. In addition, this study focuses on the thinking processes of the participants and leaves the previous studies in terms of the cognitive perspective of the process of proof. Many studies carried out were carried out only with students (Bell, 1976), only teacher candidates (Ceylan, 2012; Demiray & Işıksal-Bostan, 2017), mathematicians (Almeida, 2000), students and teachers (Samper Perry, Camargo, Sáenz-Ludlow, & Molina, 2016). This study is different from other studies in terms of sampling. Since the participants of the work were teacher and teacher candidates, it was possible to examine them from a developmental point of view. Scales, interviews and observation forms were used for data collection in poof studies. Thinking aloud protocol was generally used in the studies on metacognition. In this study, because the cognitive skills in the process of proving the cognitive skills were examined from the perspective of the study, the thinking aloud protocol, activity cards and observation form were used. Besides these, there is also a diagnostic test for proving ability. Descriptive and predictive statistics were used in the analysis of the quantitative data obtained in field studies. In the analysis of qualitative data, more descriptive analysis is performed and case are presented. The analysis of the quantitative part of the present study was done using descriptive and predictive statistics as in the field studies. In the qualitative part, content analysis was applied and the data were compared according to the participants' levels. In addition, comparative tables covering all the questions were presented. Finally, qualitative and quantitative data are correlated. However, there are studies that aim to investigate the cognitive structures of pre-service and service teachers together (Metallidou, 2009) and to study the cognitive structures of the pre-service mathematics teacher during the proving process (Barnard & Tall, 1997) in the literature. However, we detected that pre-service and service math teachers do not have enough knowledge in the literature to study the cognitive processes in the proving process. Setting cognitive situations in the proving process will also be a decisive factor for the ways to be followed in the teaching of proof (Harel & Sowder, 2007). Güler (2013) stated that it is necessary to examine the proving process in order to understand the nature of the proof. When we look at today's popular work areas, it seems that they focus on cognitive research by moving away from studies to determine opinions or knowledge-skills in areas such as problem solving, understanding mathematics (Kieran, 2017). In fact, these studies have been carried out further mentally by means of fMRI, PET, etc. (Krueger, et al., 2008; Newman, Carpenter, Varma, & Just, 2003). Such studies are important in order to expose the thinking processes of individuals (Kieran, 2017). Because when the cognitive and mental processes of the individual are known, more activities that are comprehensive can be organized and facilitated to learn. In this respect, it is important that the examination of the cognitive processes directed towards in terms of learning proof, teaching and examining the mental processes.

Barnard and Tall (1997) examined in terms of cognitive of the proving process of first year university math students. They concluded that students perform skills, which is the notion of proof by contradiction, translation from verbal to algebraic, a routinised algebraic manipulation, a link from algebra to verbal representation, synthesising a non-procedural step - correct justification, strong conviction but without justification, empirical verification, inconclusive reasoning, false reasoning, unable to respond without help- determining the way from symbols, the chance to repeat earlier arguments, establishing the contradiction. Yang (2012) studied the skills of students in reading and understanding cognitive reading, metacognitive reading, and geometry proofs by structural equations modeling in their work with secondary school students. Yang (2012) have developed a scale for this. At the scale, the skills are separated as cognitive and metacognitive. Some of the researchers' cognitive skills include reading the proposition for the first time, underlining to identify important points, drawing the shape to recognize the proposition. The researcher has metacognitive addressed the step-by-step reading of proposition, thinking about given ones, determining the key idea of proof, asking yourself how to start and ending the proof, determining the part that made the mistake, thinking and controlling what to do during the proof steps.

For this reason, the study aims to examine in order to cognitive in process of constructed algebraic proof of the pre-service and service math teacher. For this purpose, we sought to answer the following questions:

1. What kind of performed cognitive skills do pre-service and service math teachers in process of constructed algebraic proof?
2. What kind of performed metacognitive skills do pre-service and service math teachers in process of constructed algebraic proof?
3. Are there different the ability to proving the different levels pre-service and service math teachers according to their levels?

Method

Research Model

In the study, we used explanatory sequential design that it one of the mixed research designs. The explanatory sequential design is to begin with an objective quantitative study and to describe the results obtained at this stage in qualitative studies (Creswell, 2017). This design begins with the collection of quantitative data. The collected quantitative data was analyzed and the participants of the qualitative research are identified. At the next stage, we collected qualitative data and analyzed. Finally, we related quantitative and qualitative data (Creswell, 2017, p.39; Hesse-Piber, 2010). This is interesting for quantitative researchers since this pattern starts with quantitative study. However, it is the strengths of this design that it is also possible to identify participants with certain characteristics. However, this design takes too much time, as the work process requires sequential execution of quantitative and qualitative approaches. The main reasons for using explanatory sequential design in the study are; to quantitatively compare whether or not the ability to make evidence differs according to the groups, to examine the cognitive and metacognitive skills in the process of making proof by qualitative means and to examine whether the quantitative-qualitative data support each other. Another reason is to generally in the results of study, thus we conducted explanatory sequential design.

Sample

Twenty-five math teachers and forty-eight pre-service math teacher at different levels (29 senior pre-service math teacher, 19 first grade pre-service math teacher) participated in the quantitative part of the study. These participated in the qualitative part of the study of six math teachers and 12 pre-service math teacher at different levels -6 senior and 6 first grade pre-service math teachers-. Creswell and Plano-Clark (2014) stated that a two-step sample selection is more appropriate for quantitative and qualitative stages in studies using explanatory design. In the selection of the sample for the quantitative part of the study, the typical case sampling method is used. The typical case sampling method based on the selection of the average individuals to represent a case. (Gürsakal, 2013). In the qualitative part of the study, maximum diversity sampling method was used from purposeful sampling methods. The maximum diversity sampling method aims to identify different considerations in a given situation, and this type of sampling can provide a wide range of situations (Patton, 2002). Since this study aims to interpret the obtained cases on a wider sample, we used these sample selection methods respectively.

The vocational seniority of math teachers who participate in the quantitative part of the study is between 2-23 years. Seven of the teachers are female, 18 are male and 56% of the teachers graduated from Mathematics Department of Science Faculty and 44% from Mathematics Teaching Department of Faculty of Education. Fifteen of them graduated bachelor, seven of them graduated master, three of them graduated doctorate. Two teachers are in the Science High School, 2 are in the private school, 5 are in the Vocational and Technical High Schools, 6 in the "Imam-Hatip" High School and 10 in the Anatolian High School. Fifteen of the senior pre-service math teachers are female and 14 are male. 41% of senior pre-service math teacher have the experience of teaching mathematics, and 59% do not have this experience. All of the senior pre-service math teachers took the courses of Abstract Algebra and Numbers Theory I-II. Nine of the first grade pre-service math teacher are female, 10 are male. While 3 of the first grade pre-service math teachers have the experience of teaching mathematics anywhere, other prospective teachers do not have the experience of teaching anywhere. The first grade pre-service math teacher have not yet taken the courses of Abstract Algebra and Numbers Theory I-II. Proof teaching is more making these courses in math teachers programs.

In selected participants of the qualitative study, groups of achievement points were formed according to the scores obtained from the "Proving Diagnostic Test". Participants were first ranked in their own groups according to their scores in the creation of success groups (high, middle, low). Then three groups were formed by dividing the number of participants by 3. Two groups of participants were selected from the groups formed - one male, one female - were included in the qualitative part of the study. In the selection of the participants, attention was paid to the absence of participants in the group border (the lowest level of the good score group, the lowest or the top of the midpoint group, and not the top of the low score group). We have made such an application because we believe that the participants in these values will not fully reflect the characteristics of the group. Sub-category names were used for participants in direct transfers from participants. The creation of sub-category names has been made possible by aij matrix encoding. In this coding a: Teacher = T, Last grade pre-service teacher = PTL, 1st grade pre-service teacher = PT1; i: High achievement score = H, Medium achievement score = M, Low achievement score = L; j: male = 1, female = 2. For example, T_{H1}: Male with high achievement score represents teacher. Researcher was sub-category is R.

Table 1 indicated the characteristics of teachers participating in the qualitative part of the study. These variables fixed for pre-service teachers, thus we do not show in the table them.

Table 1. The Characteristics of Teachers Participating in the Qualitative Part of the Study

Participants	Faculty/ department	Level of education that graduates	Duration of undergraduate programs	Duration of experience	High-school
T_{L1}	Education/ SME	B.S.	5	6	İmam Hatip Anatolian High School
T_{L2}	Education/ SME	B.S.	5	4	Vocational and Technical Anatolia School
T_{M1}	Education/ SME	M.S.	5	12	Anatolia High School
T_{M2}	Sciences/ Mathematics	M.S. (Ph.D. Student)	4	10	Science High School
T_{H1}	Sciences/ Mathematics	B.S.	3,5	6	Anatolia High-School
T_{H2}	Education/ SME	M.S.	5	2	İmam Hatip Anatolian High School

SME: Secondary Mathematics Education

Data Collection

We used four different data collection tools in the study as "Proving Diagnostic Test", think aloud protocol, activity card and observation form. We described data collection tools in detail below.

Proving Diagnostic Test

The "Proving Diagnostic Test" used in the study was developed by Öztürk and Kaplan (2017) with secondary math teachers. In this study, we used this test because we aimed to measure the ability of mathematics teachers and teachers of mathematics to make proof. Since there is no other proof of success test in Turkish language secondary school mathematics teachers and pre-service teachers and this test includes both geometry and algebra questions, we selected "Proving Diagnostic Test" for the study. The test consists of six questions. A sample question from the geometry questions in the test is: *"'The measure of the angle formed by the intersections of the two inner bisector in a triangle is 90 more than half the size of the third angle.' Show the correctness of the proposition."* A sample question from the algebra questions in the test is: "'Each number divisible by 3 and 4 on the set of integers can be divided by 12.' show the correctness of the proposition". In the test development process, a statement table was prepared and it was determined that the content validity ratio (CVR) for each item changed between .50-1.00. The average CVR value for the 6 questions was calculated as .72. The test was applied to 80 teachers and Explanatory Factor Analysis (EFA) and item analyzes were conducted. As a result of the AFA, the test was found to be one-factor structure and explain 41.41% of the total variance. In the calculation of item discrimination, the sample was divided into upper group and lower group and the difference between them was examined by t test. As a result of the analysis made, it is determined that each substance is distinguishable. It has been determined that the item difficulty rate of the prepared test substance is within the range of .48 - .80. To provide the internal consistency of the test, the Cronbach Alpha reliability coefficient was calculated as .77. As a result of the analyzes made, it is stated that the measuring instrument is valid and reliable. For this study, Cronbach Alpha internal consistency coefficient of the test was found .65. According to Field (2009), this value is sufficient for academic achievement tests.

Think Aloud Protocol

In the think aloud protocol, we informed was informed participant about the record of the interviews, the purpose of the study, and we stated that they should voice all of their thoughts in the process. We emphasized that the participant we expected to express what he or she did and what he thought in the think aloud protocol. In the think aloud protocol, two activity cards and a semi-structured observation form were used. Firstly, we gave participants an activity card with the first proposition, and after completing the proof of the statement on this event card, we gave an activity card with the second proposition. We asked the participants to work on the activity card. At this stage, the participants' statements were recorded with a voice recorder, while the behaviors and actions of the participants were observed. We have not provided any information about the correctness or inaccuracy of the actions they have made throughout the process. To the participants, *"Is the proposition correct, why?"*, *"Is the proof you made is valid? Why, how did you decide?"*, *"Why did you do your operation?"*, *"Is there a generalization of proof you have written? Need generalization? If so, how can you generalize?"* questions such as asking questions to be fully understood.

Activity Card

Two algebra propositions have been prepared overlapping the questions in Proving Diagnostic Test the creation of the activity card. Before choosing propositions, we conducted informal interview with six secondary math teachers to determine how they should be selected using some propositions in a similar type. Two selected propositions were used as a result of the interviews. The reasons for selection and reasons for selection are explained below.

Proposition 1: "*We have a formula that gives the sum of consecutive odd numbers starting from one (n-odd numbers).*"

The main reason for choosing this proposition is that both the proposal and the proof are likely to be encountered at every stage of the educational environment. The proposition is important in terms of revealing the meaning of the concept of proof. Because the proposition can easily be proved by induction when the second side of the equation is given, it is not appropriate to apply the induction method as given in the question. Proposition does not require high level knowledge and requires basic knowledge and thinking. The proposition is to be completed with certain strategies, not the insight problem. We selected the proposition through informal interviews conducted prior to the study. In psychology studies examining the cognitive structure, problems that are generally not insightful are selected (Goldstein, 2013). Because the solution of the problems in the way of insight suddenly appears and at the same time it is not even possible to determine the skills (Goldstein, 2013).

During the informal interviews, three questions were asked to the participants. For example, *"How can you write the expression of '2+4+6+⋯+2.n' in the simplest form?", Prove that for all natural number n for which 2+4+6+⋯+2.(n-1)+2n=n.(n+1).", "Find formulate to added of even number".* We detected that the problem is usually solved by inductive method and the skills in the process can not be determined sufficiently. In order to prevent the demonstration of correctness by induction of the problem, we removed the question from the equality form. The opinions of a mathematics educator and a cognitive psychology expert have been consulted to what extent the proposal can put forward the proving process cognition. It has been determined that the question is appropriate to examine the cognitive process in view of the opinions received from the teaching members. The prepared activity card was applied with four teachers and it was determined that the language of the problem was sufficiently understood.

Proposition 2: *"Prove that let a, b be relatively prime, $a, b \in Z^+$ such that a|c and b|c, we have a.b|c."*

The question is that participants can easily recognize accuracy with intuitive or a few examples; but it is a proposition that they can have difficulty expressing symbolically. The proposition is easily progressing with algebraic operations until the latest one from the proof steps. In the last step, however, it is necessary to know another theory that is essential for divisibility so that the proof can be completed - "The two natural numbers differ from zero, each common divisor, the largest of the common divisors." The proposition can be proved using the definition of divisibility and an auxiliary theorem (lemma) rather than the internal vision problem.

The choice of this problem is based on informal negotiations conducted prior to the study. During the informal interviews, three questions were asked to the participants. For example, *"Prove that If $\forall x \in R$ and $x \neq 0$, then there is a single $\frac{z}{x} = y, y \in R$ for $\forall z \in R$" "$a \in Z^+$ and $b \in Z^+$, prove that if $a|c$ and $b|c$ then $a.b|c$" and "Each number divisible by 3 and 4 in the integer set can be divided by 12."* In informal interviews, participants were able to prove their chosen proposition, suggesting that they can see it easily and prove it; But only when they arrive at the final stage of the proof, they can not use the symbolic language they can only spell verbally. In this respect, it is thought that this problem should be included in the event card and that the situation there is necessary in order to uncover it. The opinions of a mathematics educator and a cognitive psychology professor have been consulted to determine the degree to which the proposition can put forward the process of proving cognitively. It has been determined that the question is appropriate to examine the cognitive process in view of the opinions received from the teaching members. The prepared interview form was applied to four teachers and it was determined that the language of the problem is understandable.

Observation

During the preparation of the observation form, informal interviews on the development of the activity card were utilized. Informal interviews were the first prepared unstructured observation. At this stage, the teachers were asked to take note of the skills displayed by the teachers. Observation form was created by taking notes from the notes and using the field. The observation form is written in two parts. A part of the form is designed to be likert as it examines the skills that pre-service and service teachers exhibited in the process. The other part is designed as a commentary section that will emerge in the process but will take note of the skills that are not included in the likert section. The sections are designed in triple likert type. Evaluation criteria are rated as "0, 1, 2". Participant; "0" is displayed if the specified skill is not displayed in any question, "1" is displayed in one of the questions, and "2" is displayed in the other two of the questions. When the option "1" is ticked, it is written next to the option to show the article to which question. If a description of the skills is required, the form is recorded in the description section. The observation form has been transformed into a semi-structured form in accordance with the skills demonstrated by the teachers in the process of proving.

Researcher Role

In this study the researcher is in the role of non-participant observer. The researcher asked questions that would reveal the participants' opinions in the think aloud protocol; but they have never been involved in the answers given by the participants. The researcher has not made any attempt to harmonize with the participants of the study. Participants in the study have been given permission to observe and are aware that participants are observed.

Studying Process

We was informed participation in the "Proving Diagnostic Test" about the study, and participants who were selected to have two phases of the study were required to attend the second phase. Teachers who indicated that they could not attend the second stage were excluded from the study sample. The "Proving Diagnostic Test" was conducted under the supervision of the investigator in the teachers' room of the school where the participant teachers were working and in the classroom under the supervision of the investigator. Participants were given 40 minutes for the test and no additional time for those who could not complete it.

Participants were allowed to record voice during the second session of the workshop with a think aloud protocol. All participants allowed voice recording. In the process, participants were given an activity card and asked to reveal their thoughts. The study process was carried out outside the teacher's course in the teachers' schools for the schools they were working for. Interviews with each of the teachers were held on different days. The interviews with the pre-service teachers were conducted in the guest teaching staff room of the faculty where the study was conducted. The interviews conducted with the pre-service teachers were also completed within the appropriate time frame except for the duration of the courses. Interviews conducted with pre-service teacher; was conducted in three days with first grade teacher candidates in two days with last grade teacher candidates.

Data Analysis

In the mixed method research, data analysis made according to the determined model (Creswell & Plano Clark, 2014). According to the explanatory sequential design, the data analysis should proceed as "quantitative data analysis → qualitative data analysis → associating quantitative and qualitative data" (Creswell, 2017). In this direction, firstly we concluded the analysis of quantitative data, then made the analysis of qualitative data and finally the information about the relation of quantitative and qualitative data is given. The analysis of the quantitative part of the current study was done using descriptive and inferential statistics as in the studies in the literature. In the qualitative part, content analysis was applied and the data were compared according to the levels of the participants. In addition, all the questions are presented in the comparison tables. Finally, qualitative and quantitative data have been associated.

Analysis of Quantitative Data

In the analysis of the quantitative data, we used the data collected in the diagnostic test. We applied descriptive and inferential statistics to the quantitative data in the study. As a result of descriptive statistics, the individuals participating in the quantitative study are divided into the high, medium and low groups. In this way, individuals were selected who would participate in qualitative research. The inferential statistics were used to determine the difference between the scores of teachers, senior pre-service teacher, and first grade pre-service teachers in the "Proving Diagnostic Test". For this, it was first tested whether the data provided normality assumptions. This assumes normality assumptions for both groups and for all data. For normality assumptions Kolmogorov-Smirnov test was performed, respectively, then the kurtosis and skewness values were examined, then the histogram graph was examined and finally the Q-Q and P-P graphs were examined. As a result of these operations, collected data are found to provide normality conditions. The assumptions of homogeneity of variance and independence of data were also tested and one-way ANOVA was performed because the conditions were appropriate. Among the groups with significant differences, the Games-Howell test was used when the variances were not equal. This test is the strongest among the tests used in small samples (Field, 2009).

Analysis of Qualitative Data

In the study, content analysis were used for analyzing the collected data. In the content analysis, think aloud protocol were analyzed (transcript). The data is then sub-categories by the first researchers. The resulting sub-categories are categorized according to their common characteristics. Finally, the categories are named so as to overlap with the field text, taking into consideration the features of the categories. The code-behind coding matrix is presented to a specialized teaching staff in the field of mathematics education and is required to code for 25% of the data. Then, the inter-encoder reliability [(Common answer/Total answer)x100] was used. Then the conformity between two researchers is determined as .79. According to Miles and Huberman (2015), this value is sufficient for reliability. In the sub-category not specified as inappropriate, the researcher and the expert have come together to decide whether the sub-category should be retained, modified or removed. Following the coding of the think aloud protocol, the observation forms and the activity card (document) were examined and participated in the coding of the data which could not be detected in the think aloud protocol. Categorization was created as a result of coding. The categories were created as a result of combining similar sub-categories. Finally, categorizations were grouped according to their common characteristics and cognitive skills and metacognitive skills were collected.

Related on Qualitative and Quantitative Data

There are two ways of relating quantitative and qualitative data in the study. The first of these is the selection of participants of qualitative research which is explained in the participants section. The second is comparisons of the participants according to the level of proving success and the types of metacognitive skills. At this stage, the comparison of the quantitative comparison of the achievements of the mathematics teachers, the last grade pre-service mathematics teachers and the first grade pre-service mathematics teacher.

Validity and Reliability

The present study, the sample is described in detail to provide external validity and the participant characteristics are presented in full detail. In addition to this, different data collection tools were used together in the data collection, and the opinions of the participants and the images of the activity cards were included. Three methods have been used to provide internal validity. The first of these is the methodological variation. In the study, the think aloud protocol, observation form and activity card were compared with each other. In this way, overlapping and different parts of the three data collection tools have been tried to be revealed. The second is participant validation. For this, after the coding, the participants are asked to revisit the sub-categories prepared in the think aloud protocol and to present the sub-categories to them as appropriate / not appropriate. Participants all expressed their opinion that coding is appropriate. The third is the researcher variation. For this, the data collection process has been described in detail and progress has been made by checking the entire study period to a specialized teaching member.

In order to ensure the external reliability of the work, interviews made during the data collection process were recorded with voice recorder. In the presentation of the direct transcripts, the number of the interviewed transcript and the number of the transcript is given. In order to ensure internal reliability, appropriate research model was used for the research problem, participants and data collection tools were determined in accordance with the selected research model. The collected data were analyzed according to the research problem. In addition, a mixed method research has been established to increase the generalization of the study.

Results and Discussion

Results are presented in accordance with research problems. For this, firstly, findings obtained from quantitative data were presented, then the skills displayed on the basis of cognitive skills were presented, and finally, the skills exhibited on the basis of metacognitive skills were presented.

Comparison of In-Service and Pre-Service Teachers' Proving Abilities According to Different Level Groups

In this part of the study, presentations were made according to the findings obtained from quantitative data. Descriptive statistics are presented in Table 2.

Table 2. Descriptive Statistical Results for Proving Diagnostic Test Scores

Sample	N	M	SD	Low and Upper Limitations
Teachers	25	20.28	8.58	[16.74, 23.82]
Last grade pre-service teachers	29	16.69	6.07	[14.38, 19.00]
1st grade pre-service teachers	19	11.89	3.74	[10.09, 13.70]
Total	73	16.67	7.27	[14.97, 18.37]

Table 2 show that the group with the highest average scores is teachers (M = 20.28), then final grade teacher candidates (M = 16.69) and the lowest average is first grade teacher candidates (M = 11.89). The variance analysis results are presented in Table 3.

Table 3. Results of Variance Analysis

	Sum Squares	df	Mean squares	F	p	η^2	Significant
Between group	759.073	2	379.54				
Within group	3051.036	70	43.59	8.708	.000	.45	$Ö\text{-}ÖA_1$ $ÖAS\text{-}ÖA_1$
Total	3810.110	72					

Table 3 show that the difference between the groups was significantly as a result of the one-factor variance analysis conducted to determine whether the proving diagnostic test scores differed from group to group$(F(2,70) = 8.71, p < .05, \eta^2 = .45)$. As a result of the Games-Howell test emphasized that there is a significant difference between teachers, 1st grade pre-service teachers and last grade pre-service teachers and 1st grade pre-service teachers. According to this, both the teachers 'and last grade pre-service teacher' proof test proves that the average of the achievement scores is significantly higher than the 1st grade pre-service teacher; the difference between the teachers and the final teacher candidates was not significant.

Results and Discussion on Cognitive Skills Theme

As seen in the cognitive skills theme, this theme is gathered in five categories as "Read the proposition of the proof", "Evaluating the correctness", "Determining strategy", "Carry out plans" and "Heuristic shortcuts".

First of the categories of cognitive skills theme is "Read the proposition of the proof". In this category, we detected those sub-categories: "Writes proposition symbolically", "Reading proposition for the first time", "Transforms the proposition expression into an inconsistent form", "Expresses it with its own cues to understand proposition", "Feels correctness of proposition intuitively", "He/She reads the propositions step by step", "Reads several times when it does not understand proposition and thinks about given ones", "Detected the hypothesis, the rule and aims", "Detected the key idea of proof" and "Underlying proposition to understand the hypothesis and judgment of the proof."

The skill obtained in the category of "Read the proposition of the proof" is the sub-category that "Writes proposition symbolically". From the identified participants exhibiting this skill, the solution to T_{L2}'s first question presented in Figure 1.

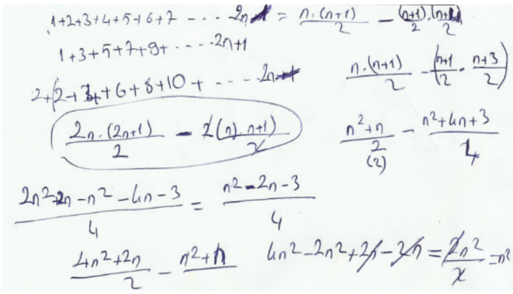

Figure 1. Screen Image Showing that T_{L2} Symbolically Expresses the Suggestion

The Figure 1 appeared that the teacher has shown the skill of "Writes proposition symbolically". Participants identified in the second proposition as symbolic writing suggest that they have exhibited the skill of the $PT1_{M1}$ as "[19.19] ... *This is probably expressed as (a, b) = 1* ... (Line, 98)". This skill is not need high-level cognitive skill. It is understood that the candidate teacher demonstrates this skill in an automated manner, not only in his own knowledge but also in the knowledge of the present. For this reason, the skill was assessed cognitively. Hanna (1995) and Fukawa-Connelly (2012) emphasized in the context of information use symbolic expressions for formal proof is cognitive skill. Therefore, we can say that interpreted skill of "Writes proposition symbolically" as cognitive skill confirms earlier findings.

Another cognitive skill that is reached in this category is the sub-category "Reading proposition for the first time". This sub-category was unearthed when the participants read the whole thing without hesitation. The participant expressions obtained for this skill are identical. To give an example, we can say that the expression "[04.20] *We have a formula that gives the sum of odd-number consecutive odd-numbered numbers starting from one to n up to n* ... (Line, 32-33)" of T_{L1} supports this situation. "Reading proposition for the first time" when dealt with we detected in the present study that this skill is not need high-level cognitive skill. Therefore, this sub-category was interpreted as cognitive skill. There are studies that indicate that this skill is a cognitive skill in the literature (McKeown & Beck, 2009; Yang, 2012). In this context, it can be stated that skill of "Reading proposition for the first time" was cognitive skill. This result confirms earlier findings.

Another cognitive skill that is detected in the category of "Read the proposition of the proof" is the sub-category "Transforms the proposition expression into an inconsistent form." It has been determined that this skill is exhibited only by the T_{M2} participant in the first proposition. The participant "[18.30] ... *1-3-5-7 and n is the odd number. We have a formula that gives the sum of consecutive odd numbers.* (Line 104) has been found to exhibit this skill from the expressions "Transforms the proposition expression into an inconsistent form". The skill was transformed proposition more complex structure. In this context, this skill was evaluated as cognitive skill. Bruning et al. (2014) mentioned transformed inconsistent form while problem solving can be difficult and caused issue with language. In this regard interpreted of skill of "Transforms the proposition expression into an inconsistent form" as cognitive skill confirms their study.

Another skill that is detected in the category of "Read the proposition of the proof" is "Expresses it with its own cues to understand proposition". Participants who were determined to exhibit this skill expressed his opinion with the expressions of PTL_{L2} "[27.21] *Expressing their desire for the sum of consecutive natural numbers counted from one to the other ...* (Lines, 151-152). This sub-category is only found at the beginning of the proposition. The skill of "Expresses it with its own cues to understand proposition" is not need high-level activity. Thus, it was interpreted as cognitive skill. Another cognitive skill that is reached in the relevant category is the sub-category, "Feels correctness of proposition intuitively." PTL_{L2}, which is determined to exhibit this skill, which is determined to be exhibited only on the first proposal, in the continuation of the skill determined in the previous sub-category. "[27.21] *... I want to do a few tests* (Makes a few trials) *... the square of 1, the square of 2, the square of 3 will continue. It will obviously continue in the following. So the term will continue to be the square of the number. I feel like that.* (Line, 156-158)" is understood to show this skill. Many studies emphasized this skill is metacognitive skill (Aydemir & Kubanç, 2014; Cozza & Oreshkina, 2013; Schraw & Dennison, 1994). This result conflicts earlier findings. The reason for this, a skill was cognitive for an individual, it maybe metacognitive for another individual. This reason, although this skill detected as cognitive skill earlier research, we detected it as metacognitive skill in the present study.

Participants identified as exhibiting "He/She reads the propositions step by step" sub-category were found to display this sub-category from the following expressions of PTL_{L1}. "[30.39] *If a and b are positive integers and prime between them ...* (Line, 197), [30.49] *... If a and b are prime ...* (Line, 197), [30.49] *... dividing a, c ...* (Line, 200), [31.39] *... if c divides b ...* (Line, 203), [32.49] *... does ab, c divide* $2k^2$ *? It says ...* (Line, 212)". Another skill was sub-category of "He/She reads the propositions step by step". As seen in expression of participants, this although this skill requires awareness, it has been automated in this study. Therefore, we interpreted this sub-category as cognitive skill.

Another cognitive skill reached in the category of "Read the proposition of the proof" is the sub-category "Reads several times when it does not understand proposition and thinks about given ones". PTL_{H1} of participants determined to exhibit this skill "[10.49] *Starting from 1 to n, the odd number starts from 1 ... 1 + 3 + ... + n n is the last odd number ...* (Line, 60-61), [14.28] *... from 1 to the sum of the numbers up to the number n ...* (Line, 77)." The skill of "Reads several times when it does not understand proposition and thinks about given ones" was not needed high-level thinking. Participants repeat read because of lack of attention or they thinking a short time for not understand. Thus, we interpreted it as cognitive skill. Yang (2012) founded that thinking about given is metacognitive skill as planning. Evaluated as cognitive of this skill result conflicts findings of Yang (2012). However, we detected that this skill was not supra-memory activity. Therefore, we interpreted that skill of "Reads several times when it does not understand proposition and thinks about given ones" was cognitive skill. Upper memory activity is the ability of the individual to be aware of the information and operation of his or her own memory and to use the information by manipulating it for the purpose of operation. It is a process involving three stages of recall, awareness and recall (Irak, 2005).

Another cognitive skill that is identified in this category is the sub-category "Detected the hypothesis, the rule and aims". The opinions of some of participants performed this skill are presented in Table 4.

Table 4. The statements of some of the identified participants demonstrating the skill of "Detected the hypothesis, the rule and aims"

Participant	Proposition	Expressions
PT_{L2}	1	"[27.21] ... 1 is a consecutive number; 1, 3, 5, 7, 9, 11, 13 and finally is n. Here n is odd number. Now we have to find the formula that gives this sum. It would be good to find the final step of the operating step in front of my eyes. (Line, 152-154)"
T_{M2}	2	"[24.14]... I'm going to show you the following: a.b. Will c split? So if I can show that c = a.b.m, I can show that at least one m is an integer of m number if the equation is true… (Line, 145-147)"

Table 4 indicated that both propositions have participants identified as "Detected the hypothesis, the rule and aims." This skill generally made automatically and it is not high-level thinking skill for formal proof. Therefore, we interpreted it as cognitive skill. Kaplan and Duran (2015) evaluated as metacognitive skill for middle school students determined skill purpose and semi-purpose. Evaluated as cognitive of this skill result conflicts findings of them. The reason of this different maybe from sampling. In other words, this skill can be metacognitive skill for middle school students, cognitive skill for pre-service and service teachers.

Another cognitive skill that is reached in the relevant category is "Detected the key idea of proof." The dialogue between T_{L2} and the researcher, determined to exhibit among the determined participants exhibiting this skill, is as Figure 2 (Line 55-59).

[15.55] TL2: a divides c and b divides c. Implies a.b divides c. Is not it? ... Since a and b are prime number, their greatest common divisors (gcd) are equal to the number of 1 least common multiple (lcm).
[17.25] R: Why did you move from greatest common divisors and least common multiple?
[17.28] TL2: When you gave them prime number, greatest common divisors and least common multiple called me. When we get out of here, gcd (a,b) are multiplied by 1 and gcd (a,b) are multiplied by numbers. If a divides c, b divides c, c is the common multiple of these numbers.

Figure 2. Dialogue is between on T_{L2} and R

When the dialogue is examined, it is understood that the teacher identifies the concepts of greatest common divisors and least common multiple as key proofs of proof. This situation is interpreted as "Detected the key idea of proof." We detected it as cognitive skill in the present study. Similarity to Şahin (2016) reveal that pre-service math teachers determined key idea for proof while made divisible proof. Raman (2003) reported that determining key idea of proof was intuitive. In this context, interpreted as cognitive of skill of "Detected the key idea of proof" confirm study of Raman (2003).

The last skill to be reached in the category of reading propositions is the sub-category "Underlying proposition to understand the hypothesis and judgment of the proof." The dialogue between T_{H2} and the researcher (Fig 3) is determined by participants who have demonstrated this skill (Line, 79-83).

[18.43] R: In this question, you underlined proposition. Why?
[18.45] T_{H2}: I have rounded the end again. I'm doing something to tell you what you want from us. The reason for this is that the number n is one number, so to understand what the questions are.
[19.02] R: So the place where you put the round is a more important place, the place where you underlying is an important place?
[19.04] T_{H2}: So it can be said. But I think more like hypothesis and provision.

Figure 3. Dialogue is between on T_{H2} and R

When dialogue is examined, it is understood that the participant exhibits his / her skill as "Underlying proposition to understand the hypothesis and judgment of the proof". Because of participants performed this skill as automatically, this skill interpreted as cognitive skill. Erdem (2005) point out its underlying while reading test is important, but if this skill made as automatically, he/she can be underlying unnecessary section. We detected participants performed this skill as automatically. Thus, we interpreted this skill as cognitive skill. This result confirm earlier findings.

Another category reached in the study is the category "Evaluating the correctness". In this category, "Evaluates the correctness of the proof according to the ritual proof scheme", "Evaluates the correctness as intuitive", "Asks the researcher the correctness of written operations in the proving process", "Evaluates the correctness of the proof according to the authoritarian proof scheme", "Evaluates correctness as inductive proof scheme", "Evaluates correctness by checking operations" and "Evaluates correctness of proof by performing all operations again." This skill is not ensure validity of mathematical proof.

First of sub-categorys in this category was "Evaluates the correctness of the proof according to the ritual proof scheme". Participant performed of this skill who is T_{L2} emphasized "[21.15] Proof is correct, but I don't know it is validity. Experts know its validity (Line, 69)" and PTL_{L2} emphasized "[42.41] Proof was not validity. In other words I show that correctness as a symbolic, but maybe I don't written this as a mathematical (Line, 202-203)." This sentences show that participants performed "Evaluates the correctness of the proof according to the ritual proof scheme". Individuals made justification in according to ritual proof scheme were unstable. In other words, they self-indulgent about correctness of proof. Many studies show that participants performed this skill (Doruk & Kaplan, 2015; Doruk, 2016; Harel & Sowder, 1998; Martin & Harel, 1989). Determined of skill of "Evaluates the correctness of the proof according to the ritual proof scheme" confirm earlier findings.

Another skill in this category was sub-category of "Evaluates the correctness as intuitive". The opinions of some of participants performed this skill are presented in Table 5.

Table 5. The statements of some of the identified participants demonstrating the skill of "Evaluates the correctness as intuitive"

Participant	Proposition	Expressions
PTL_{M1}	1	"[39.46] I understood that two prime numbers divides one number implies their multiple divides it number. Indeed if I written $(x+1)$ instead of k, c multiple $a.b$. Therewithal f multiple x, therefore c multiple 'a.b'. However, I don't put on paper." (Line, 98-101)
T_{M1}	2	"[39.46] If a and b divides c, and b relatively prime, only if a and b multiplier c. Thus if c have got two multiplier as a and b, $a.b$ equality to c. However, I must be proving this expression." (Line, 105-107)

As seen in Table 5, examining participants' expressions we can interpreted participants performed "Evaluates the correctness as intuitive" skill. In addition to it is not need high-level thinking skill and it based only on heuristic. Thus, we interpreted it as cognitive skill. Samper et al. (2016) found that participants made proof as heuristics. MacDonald (1973) noted intuitive proof is not as easy as taking the memorandum. This proof method was instructor, but it is not proof in reality.

Another cognitive skill in this category was sub-category of "Asks the researcher the correctness of written operations in the proving process". For instance, T_{H2} answered Proposition 1-2 as follows:

> "[18.18] I understood correctness, isn't it? Consecutive odd numbers up to n... starting from 1. (Line, 78)." (Proposition 1)
>
> "[32.07] Will be the last (k.t)/c? Is it true? (Line, 135)" (Proposition 2)

This skill is not need self-evaluated and made questioning. Therefore, it evaluated as cognitive skill. Similarity to Şahin (2016) found that pre-service math teacher questioning researcher correctness of proving process. Dialogue between the researcher and the participant in the process of the study asks the investigator to verify that the participant is the investigator as the authority. In other words, the participant has gone to an authority here (this authority can be a teacher sometimes and sometimes books, etc.). If the student had directed the same question to himself, he could have been accepted as a metacognitive, considering that the participant was questioning. However, this skill is cognitively categorized as it is understood that the participant has tried to confirm to an authority.

Another sub-category was "Evaluates the correctness of the proof according to the authoritarian proof scheme". This skill was detected only first proposition. T_{L1}'s answer to illustrate it. He said, "[11.05] *If my proof is validity, I must be proving to correctness total of numbers from 1 to n. I think don't problem because of based on theorem* (Line, 48-49)." We interpreted these sentences as "Evaluates the correctness of the proof according to the authoritarian proof scheme". This skill is not need high-level thinking. It is evaluated based on authoritarian. Therefore, we interpreted it as cognitive. In her study, in which he investigated the proof schemes of pre-service middle mathematics teachers in Çontay (2017), he determined that the teacher candidates used authoritarian proof scheme when making proof. The skill of evaluating the correctness of the proof according to the authoritarian proof scheme is considered to be cognitive skill since it does not require the higher level reasoning skills of the person and requires an authority-based evaluation.

"Evaluates correctness as inductive proof scheme" was a another sub-category. As seen in $PT1_{M2}$'s answer, performed this skill can be achieved. She said that "[00.51] *When I written only the formula, it isn't proof.* (Line, 6) [00.58] *… I can prove it by value.* (Line, 8). [01.02] *… For example, it said that from 1 to n. If I try to 1, it is 1 (2-1). If I try to 2, it is 3 (4-1). If I try to 15, it is 29 (30-1). It happens this way.* (Line, 10-11)." This result confirm earlier findings. Smith and Kosslyn (2014) noted inductive reasoning is not give exact result about correctness of information. It provide foresight as generally. Goldstein (2013) emphasized inductive reasoning is not awareness performed skill as generally. Martin and Harel (1989) found that pre-service teachers performed this skill overcharged. This result confirm earlier findings.

Another cognitive skill that is determined in this category is the sub-category "Evaluates correctness by checking operations". The dialogue between T_{L1} and the researcher (Fig. 4) is determined by participants who have demonstrated this skill (Line, 39-41).

[06.41] T_{L1}: … [He has understood operations as made] I did not find the formula.
[08.05] R: Why are you think like that?
[08.11] T_{L1}: Because this operation wasn't correctness. Maybe this formula can be equality it.... [He made operations again]

Figure 4. Dialogue is between on T_{L1} and R

As shown in Fig. 4, the teacher performed skill of "Evaluates correctness by checking operations". Another cognitive skill was the sub-category of "Evaluates correctness by checking operations". Participants who performed this skill controlling operations and evaluated whether true or not as automatically. Öztürk, Akkan, and Kaplan (2014) found gifted students controlling operations in evaluated process of problem solving. They emphasized this skill is metacognitive. Different assessments of skills may be due to different sampling characteristics.

The last skill to be reached in the category of "Evaluating the correctness" is the sub-category of "Evaluates correctness of proof by performing all operations again." PTL_{H2}'s answer is depicted following (Line, 65-69):

"[18.58] I controlled my operations again. I called as a which is total of numbers from 1 to $2t + 1$ that it is any odd number. I called as b which is even numbers in this rank. I bought 2 parentheses to b. I hypothesize that I know total of numbers from 1 to n. In that case b is $t^2 + t$. I substituted total of $a + b$ and then I reached that a is $(t + 1)^2$. Firstly I had written n was $2t + 1$. Thus $t + 1$ was $(n + 1)/2$. I reached again my transformers."

These expressions indicated that pre-service teacher performed skill of "Evaluates correctness of proof by performing all operations again". This skill is not need high-level thinking skill. Thus, we interpreted it as cognitive skill. Özkaya and İşleyen (2012) found pre-service math teachers made operations again while problem solving in theme of functions. So they evaluated whether true or not. This result confirms earlier findings.

Another category reached in theme of cognitive skills was the category of "Determining strategy". In this category, we have reached the following cognitive skills: "Verbally expresses the proof of proposition", "Proving in the way it memorizes" and "He/she has low self-efficacy in applying the proof strategy."

The first skill to be reached in the category of "Determining strategy" is the sub-category of "Verbally expresses the proof of proposition." $PT1_{M1}$'s answer is depicted following (Line, 37-41):

"[10.19] For example, let's accepts it was 1+3+5+...+n. Number of terms was (last term-first term)/ amount of increase. Amount of increase is 2. For example, if we bring this and make half of the sum of the first term with the last term, it was (n+1)/2. We are doing the same here. We collect the first term in the last term, but it is half the total number of terms. Then divide it by 2 so maybe I can get the formula like this."

When the expressions of participants are examined, it is understood that they exhibit the skill of "Verbally expresses the proof of proposition." T_{L2} written the data shown in Fig. 5 for proposition 2.

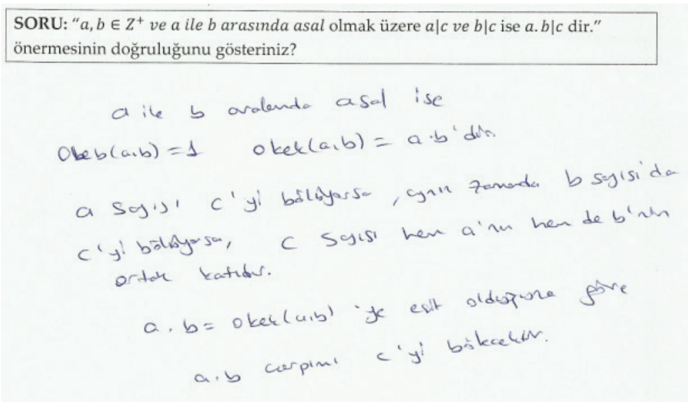

Figure 5. The skill of "Verbally expresses the proof of proposition." (T_{L2} written the data)

Fig 5 show that the participant performed the skill of "Verbally expresses the proof of proposition." Another category reached in theme of cognitive skills was the category of "Determining strategy". This skill interpreted as cognitive skill, because of it is not important for correctness of mathematical proof. Many studies show that participants performed this skill (Nool, 2012; Şahin, 2016). In this context, we can said that the results confirm earlier findings.

Another cognitive skill that is reached in this category is "Proving in the way it memorizes." The statements of some of the participants identified as exhibiting this skill are presented in Table 6.

Table 6. The statements of some of the identified participants demonstrating the skill of "Proving in the way it memorizes"

Participant	Proposition	Expressions
T_{L2}	1	"[13.29] There was a formula that gave the sum of consecutive odd numbers. Sum of terms = number of terms. The median term... The number of term is (Last term- first term)/ (amount of increase+1). Other is (Last term+1)/2. The results of these two operations is multiplied. In other words this formula is square of (n+1)/2. (Line, 48-50)"
$PT1_{L2}$	1	"[26.21] …Starting from 1 and consequent odd number. Thus I aided 2 for even term. It was reached n as 1+3+5+7. This formula is based on the recipe. I do not remember the formula. (Line, 198-199)"
PTL_{H1}	2	"[24.39] We are already showing her presence here; But I do not know it is very accepting. It was a nice surprise; But the moment does not come to mind... (Line, 115-116)"

Table 6 show that the participants exhibit the skill of "Proving in the way it memorizes." Another cognitive skill was sub-category of "Proving in the way it memorizes". This skill based on memory of individual. Therefore participants done as much as they can remember. We interpreted it as cognitive skill with this reason. A few study show that participants performed this skill (Şahin, 2016).

The last cognitive skill in the category of "Determining strategy" was sub-category of "He/she has low self-efficacy in applying the proof strategy." PTL_{M1}'s expressions indicated his performed this skill. He said that "[27.04] *I can apply proofs by contrapositive, but I don't believed that I will solve it.* (line, 77-78)." This sentences show that pre-service teacher performed skill of "He/she has low self-efficacy in applying the proof strategy." Self-efficacy ensures that you are engaged for a longer period of time and work longer if you can not solve a problem in mathematics (Bruning et al., 2014). Because this skill has low self-efficacy, it will shorten the length of time that individuals are engaged in proposals or are concerned with proposals if they cannot. For this reason, low self-efficacy was assessed as cognitive skill without metacognitive activity.

The category of "Carry out plans" was another category in the theme of cognitive skills. In this category, pre-service and service teachers performed those sub-category: "He/she considered that lemma is proved in proving process", "He/she made pattern generalization" and "He/she refuses to give examples by saying that he is not right".

The sub-category of "He/she considered that lemma is proved in proving process" was first sub-category in this category. The dialogue between T_{H2} and the researcher (Fig. 6) is determined by participants who have demonstrated this skill (Line, 100-104).

[22.40] T_{H2}: I think this solve isn't the proof.
[22.41] R: Why?
[22.42] T_{H2}: Because I starting with an admission. Indeed, while proving us sometimes admission to postulate. We made admission, but I must be explained this. In other words, I starting by knowing.

Figure 6. Dialogue is between on TL1 and R

When examining the dialog, we understood that teacher considered lemma is proving for every proof. This case show that teacher performed skill of "He/she considered that lemma is proved in proving process". This skill is an idea that expresses the opinions of individuals on the concept of proof and has been evaluated as metacognitive since it does not require supra-memory activity.

Another sub-category in category of "Carry out plans" was skill of "He/she made pattern generalization". This sub-category detected only for proposition 1. PTL_{L1}'s expressions show that pre-service teacher performed this skill.

"[28.16] …I will do this formula. So what is the rationale first, for example, the sum of the first four terms? I will use as a term 16.n and I generalization formula for 16. I don't memorized. What should I do? Now, because for $n = 1$ are equal to 1. Because for $n = 2$ are equal to 4. Because for $n = 3$ are equal to 9. Because for $n = 4$ are equal to 16. We find n^2. Thus, result equal to n^2." (Line, 183-186)

Another skill in this category was the sub-category of "He/she made pattern generalization". Pattern generalization is not proof. In other words, pattern generalization does not guarantee correctness of proof. Thus, we interpreted this skill as cognitive. Čadež and Kolar (2015) noted generalization is the deductive reasoning carried out in formal ways that do not require experience. In this sense, it can be said that the determination of pattern generalization as cognitive skill is supported by the mentioned studies.

The last skill in the category of "Carry out plans" was skill of "He/she refuses to give examples by saying that he is not right". This sub-category detected only for proposition 2. $PT1_{L2}$'s expressions show that pre-service teacher performed this skill.

"[19.14] For example, let a and b be relatively prime numbers. Obtain a and b odd numbers and c even number. This is not corrected. I assumed that a equality to c. It is enough to give one non-example." (Line, 64-66)

When examining the expression of pre-service teacher, we understood that PTL_{L2} performed skill of "He/she refuses to give examples by saying that he is not right". This skill evaluated as cognitive skill in the present study. Because the proposals involved in the activities given to participants in the study are correct and their correctness needs to be shown. Participants determined to exhibit this skill often misunderstood or misrepresented the proposition. Güler and Ekmekci (2016) found pre-service mathematics teachers offer weak justifications when refusing to proposition. This result confirm earlier findings.

The last category in the theme of "Cognitive Skills" was category of "Heuristic shortcuts". The sub-categories of "Always advances towards purpose", "Analyzed instrument and purpose", "Climb the hill" and "Searching random" were collected this category.

The sub-category of "Always advances towards purpose" the first cognitive skill in this category. This sub-category detected only for proposition 2. T_{L1}'s expressions show that pre-service teacher performed this skill (See. Fig. 7).

"[14.08] Let $a.k$ equality to $b.t$ and a, b relatively prime numbers. a is not equality to b. If multiple of these numbers equality, k and t were relatively prime numbers. The numbers relatively prime number and equality to multiple, thus I can say that b equality to k and a equality to t." (Line, 61-64)

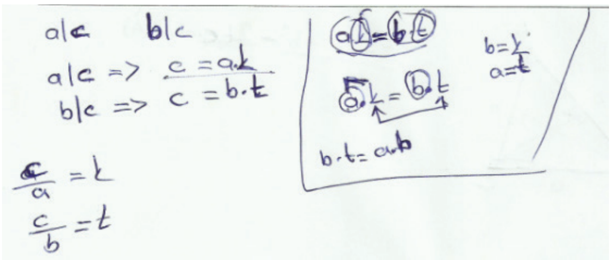

Figure 7. The skill of "Always advances towards purpose." (T_{L1} written the data)

These sentences indicated to teacher performed skill of "Always advances towards purpose". Activity card of same teacher the data in Fig. 7. This case indicated to teacher performed skill of "Always advances towards purpose". In other words, the solution of some problems, especially in the field of algebra, can not be achieved directly by simplification. After a complexification process is performed, simplification can be started. In such problems, if you go directly to the result when you see the target, the result will usually not be reached (Arcavi, Drijvers, & Stacey, 2017).

Another cognitive skill that is reached in this category is " Analyzed instrument and purpose." The statements of some of the participants identified as exhibiting this skill are presented in Table 7.

Table 7. The statements of some of the identified participants performed the skill of "Analyzed instrument and purpose"

Participant	Proposition	Expressions
T_{L1}	1	"[09.45] I made proof again. From $(4n^2 + 2n){:}\,2 - n$ I deduce that $4n^2 - 2n^2 + 2n - 2n$. After expanding and rearranging I obtained $(2n^2{:}\,2).n^2$. I take from 1 to n. Then because for finding for total of odd numbers subtraction from 1 to 2n. Therefore I has n^2. Proof is finished. (Line, 43-46)."
PTL_{M1}	2	"[34.22] …When we divide this as $x.(x + 1)$ divide one x. We arrived this question -Is $(x + 1)$ divided k?-. $(x + 1)$ equality to b. We arrived this question -Is b divided k? Indeed, b common multiple of k. So I understand it from here (Line, 91-93)."

When Table 7 indicated the participants try to analyze by doing process in the process. However, the transactions performed by the participants are actions that will not guarantee the correctness of their results. Approaches in which certain algorithms are used, but these algorithms do not guarantee to reach the end are called vehicle-objective analysis. The finding obtained for this reason is interpreted as exhibiting the participant's " Analyzed instrument and purpose."

Another cognitive skill that is reached in this category is the "Climb the hill." Some examples of participants identified as exhibiting this skill are presented in Table 8 as an example.

Table 8. The statements of some of the identified participants performed the skill of "Climb the hill"

Participant	Proposition	Expressions
$PT1_{L1}$	1	"[11.27] No, I don't agree with this. This solve was firstly recall from my mind, thus I made this." (Line, 41).
T_{H2}	2	"[27.18] …I will controlling whether integer or not. Let's put a question mark here. Suppose that $c/[(c/k).(c/t)]$. Therefore $(k.t)/c$. If replace c with $a.k$, k was dissolved and t/a stay…" (Line, 127-128).

Table 8 show that the participants are approaching the correct solution, but they can not reach the correct conclusion (they use shortcuts). The closest wrong solution to the right solution is called peak climbing. For this reason, it was understood that the participants exhibited the skill of "Climb the hill." (Smith & Kosslyn, 2014, p. 417).

The last cognitive skill in this category was sub-category of "Searching random". We detected perform of this skill with observations for Proposition 1. T_{H2} performed this skill for Proposition 2. The teacher said that "[34.00] *Can not I write a in b? This time, instead of k, I'll change a little more how will look?* $a.c$ *instead of writing c / b...* (Line, 142-143)." PTL_{L2} who performed this skill for Proposition 2 written the data Fig. 8.

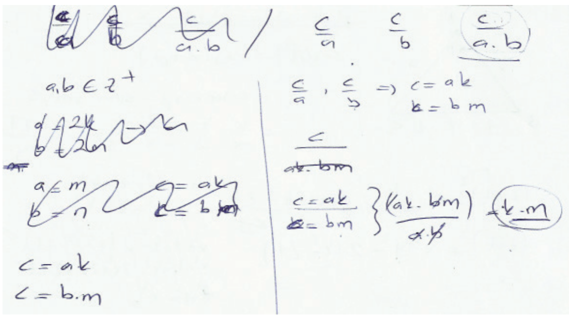

Figure 8. The skill of "Searching random." (PTL_{L2} written the data)

Fig 8 show that the participant performed the skill of "Verbally expresses the proof of proposition."

The last category in the theme of "Cognitive Skills" was category of "Heuristic shortcuts". The sub-categories of "Always advances towards purpose", "Analyzed instrument and purpose", "Climb the hill" and "Searching random" were collected this category. It has been determined that participants identified by these skills are trying to reach the shortest path. These skills can cause the participants to go to the destination in a short way when they see it as a target, which may cause the actions to be done in the meantime to be ignored and done wrongly (Smith & Kosslyn, 2014, p. 417). This result confirm earlier findings.

Table 9 indicated that distribution in according to propositions and participants of the sub-category of theme of "Cognitive skills".

Table 9. Distribution in according to propositions and participants of the sub-category of theme of "Cognitive skills".

Category	Sub-category	Propositions		Performed skill teachers		Performed skill last grade pre-service teachers		Performed skill 1st grade pre-service teachers	
		f	No	f	Teachers	f	Pre-service teachers	f	Pre-service teachers
Read the proposition of the proof	Writes proposition symbolically	2	1.S	4	T_{L2}- T_{M1}- T_{M2}- T_{H2}	3	PTL_{L2}- PTL_{M2}- PTL_{H2}	6	$PT1_{L1}$- $PT1_{L2}$- $PT1_{M1}$- $PT1_{M2}$- $PT1_{H1}$- $PT1_{H2}$
			2.S	4	T_{M1}- T_{M2}- T_{H1}- T_{H2}	2	PTL_{L2}- PTL_{H1}	-	-
	Reading proposition for the first time	2	1.S	5	T_{L1}- T_{L2}- T_{M1}- T_{H1}- T_{H2}	3	PTL_{M2}- PTL_{H1}- PTL_{H2}	6	$PT1_{L1}$- $PT1_{L2}$- $PT1_{M1}$- $PT1_{M2}$- $PT1_{H1}$- $PT1_{H2}$
			2.S	4	T_{L1}- T_{L2}- T_{M1}- T_{H1}	2	PTL_{M2}- PTL_{H1}	-	-
	Transforms the proposition expression into an inconsistent form	1	1.S	1	T_{M2}	-	-	-	-
	Expresses it with its own cues to understand proposition	1	1.S	-	-	1	PTL_{L2}	-	-
	Feels correctness of proposition intuitively	1	1.S	-	-	1	PTL_{L2}	-	-
	He/She reads the propositions step by step	2	1.S	1	T_{M2}	2	PTL_{L1}- PTL_{M1}	-	-
			2.S	1	T_{M2}	2	PTL_{L1}- PTL_{M1}	-	-
	Reads several times when it does not understand proposition and thinks about given ones	2	1.S	1	T_{M1}	2	PTL_{M1}- PTL_{H1}	-	-
			2.S	1	T_{M1}	2	PTL_{M1}- PTL_{H1}	1	$PT1_{L2}$
	Detected the hypothesis, the rule and aims	2	1.S	-	-	2	PTL_{L2}- PTL_{H1}	-	-
			2.S	3	T_{L2}- T_{M2}- T_{H2}	3	PTL_{L1}- PTL_{H1}- PTL_{H2}	-	-
	Detected the key idea of proof	1	2.S	3	T_{L2}- T_{M1}- T_{M2}	1	PTL_{H2}	-	-
	Underlying proposition to understand the hypothesis and judgment of the proof	2	1.S	1	T_{H1}	-	-	-	-
			2.S	1	T_{M2}	-	-	-	-
Evaluating the correctness	Evaluates the correctness of the proof according to the ritual proof scheme	2	1.S	-	-	-	-	1	$PT1_{M2}$
			2.S	1	T_{L2}	2	PTL_{L1}- PTL_{L2}	-	-
	Evaluates the correctness as intuitive	2	1.S	1	T_{M2}	2	PTL_{M1}- PTL_{H2}	-	-
			2.S	4	T_{L1}- T_{M2}- T_{H2}	-	-	2	$PT1_{M2}$- $PT1_{H1}$
	Evaluates the correctness of the proof according to the authoritarian proof scheme	1	1.S	3	T_{L1}- T_{M2}- T_{H2}	-	-	1	$PT1_{H2}$
	Evaluates correctness as inductive proof scheme	1	1.S	-	-	-	-	2	$PT1_{L1}$- $PT1_{M2}$
	Asks the researcher the correctness of written operations in the proving process	2	1.S	1	T_{H2}	2	PTL_{M1}- PTL_{H1}	-	-
			2.S	1	T_{H2}	3	PTL_{L1}- PTL_{H1}- PTL_{H2}	-	-
	Evaluates correctness by checking operations	1	1.S	1	T_{L1}	-	-	-	-
	Evaluates correctness of proof by performing all operations again	1	1.S	-	-	1	PTL_{H2}	-	-

Table 9. Continued

Category	Sub-category	Propositions		Performed skill teachers		Performed skill last grade pre-service teachers		Performed skill 1st grade pre-service teachers	
		f	No	f	Teachers	f	Pre-service teachers	f	Pre-service teachers
Determining strategy	Verbally expresses the proof of proposition	2	1.S	-	-	-	-	1	ÖA1$_{O1}$
			2.S	1	Ö$_{D2}$	-	-	-	-
	Proving in the way it memorizes	2	1.S	2	Ö$_{D1}$- Ö$_{D2}$	2	ÖAS$_{i1}$- ÖAS$_{i2}$	1	ÖA1$_{i2}$
			2.S	-	-	1	ÖAS$_{i1}$	1	ÖA1$_{O2}$
	He/she has low self-efficacy in applying the proof strategy	1	2.S	-	-	1	ÖAS$_{O1}$	-	-
Carry out plans	He/she considered that lemma is proved in proving process	1	1.S	1	Ö$_{i2}$	-	-	-	-
	He/she made pattern generalization	1	1.S	-	-	3	ÖAS$_{D1}$- ÖAS$_{D2}$-ÖAS$_{O2}$	-	-
	He/she refuses to give examples by saying that he is not right	1	2.S	-	-	-	-	2	ÖA1$_{D2}$- ÖA1$_{i1}$
Heuristic shortcuts	Always advances towards purpose	1	2.S	2	Ö$_{D2}$-Ö$_{O2}$	1	ÖAS$_{i1}$	-	-
	Analyzed instrument and purpose	2	1.S	1	Ö$_{D1}$	1	ÖAS$_{O1}$	-	-
			2.S	-	-	1	ÖAS$_{O1}$	-	-
	Climb the hill	2	1.S	-	-	-	-	1	ÖA1$_{i2}$
			2.S	1	Ö$_{i2}$	2	ÖAS$_{D2}$- ÖAS$_{i2}$	-	-
	Searching random	2	1.S	1	Ö$_{D1}$	-	-	-	-
			2.S	1	Ö$_{i2}$	2	ÖAS$_{D2}$- ÖAS$_{O1}$	-	-

Table 9 show that the most performed skill was "Reading proposition for the first time" in theme of cognitive skills. The least performed skills were "Transforms the proposition expression into an inconsistent form", "Expresses it with its own cues to understand proposition", "Feels correctness of proposition intuitively", "Evaluates correctness by checking operations", "Evaluates correctness of proof by performing all operations again" and "He/she considered that lemma is proved in proving process". In the context of categories, the category which is determined to be the most frequently exhibited category is the category of "Read the proposition of the proof", while the least exhibited category is the "Carry out plans" category. The skills that teachers most performed "Reading proposition for the first time", pre-service teachers most performed "Reading proposition for the first time" and "Writes proposition symbolically". In the theme of cognitive skills, teachers showed the greatest number of skills, while the least skill was in the first grade pre-service math teacher.

Results on Metacognitive Skills Theme

When examining metacognitive skills theme, this theme is gathered in eight categories as "Facilitation the operations", "Questioning", "Awareness", "Planning", "Strategy determination", "Controlling", "Relationship" and "Analogical reasoning".

We detected three sub-categories in the category of "Facilitation the operations". These sub-categories were "Change of variable for facilitation the operations", "Avoid fractional expressions for facilitation the operations" and "Detected key idea for proof".

The first metacognitive skill in category of "Facilitation the operations" was "Change of variable for facilitation the operations". T_{H1} was one of participants who performed this skill. For example, T_{H1} said that:

> "[18.49] Since we can express n easily here, I have defined an k-number myself. Question say that up to n, but it must be an odd number. We used k-number to make it easier to express the odd number here. If I turn this to n, the number of terms (k + 1) will still appear. After all, nothing will change. Here, (2n + 1) as he said because he did not say the number n as a single number. But we could express it more easily in a single-number format. I did not recognize anything new. I'm going to write n numbers here now, but looking at it maybe it's the only one that will forget. 2k + 1 or 2k-1 seems to be clearer in odd number format. We can be transformed n-number. We take the k-number and replace it. For example, if we take k-number, it was (n-1):2. What comes from here? We written replace k with (n-1):2. What comes from here? I guess I just said that I said it. In other words, it was (last term + first term):2. It was square of mean. Since we are arithmetic, we will end up with the first term from the beginning, the last term from the beginning, and the previous term from the second term. The same thing will happen if we connect n-number." (Line 124-136)

When examining sentences of the teacher, we interpreted the case as performed skill of "Change of variable for facilitation the operations" of teacher. Because T_{H1} made change of variable and he explained this case made for facilitation the operations.

Another metacognitive skill in this category was sub-category of "Avoid fractional expressions for facilitation the operations". PTL_{L1}, one of participants performed this skill, said that "[31.39] *Let replace a with $2k$. If c divisible a, c equality to $4k$. If c divisible b and c equality to $4k$, then b equality to $4k$...* (Line, 203-204)". Then research asked that "[32.47] *Why did you use 2k, 4k expressions?* (Line, 209)". Teacher answer, "[32.49] *Replace not with 2. For example, if k equality to five, it was don't. In other words, I must be combine in the common multiple in divisibility questions…* (Line, 210-211)". This dialogue indicated that pre-service teacher performed skill of "Avoid fractional expressions for facilitation the operations".

The last sub-category in the category of "Facilitation the operations" was skill of "Detected key idea for proof". This sub-category performed only for Proposition 2. PTL_{H1} was one of the participant performed in this skill. For example, he said that "[19.19] *… If this numbers relatively prime numbers,* $lcm(a, b)$ *is 1. No,* $gcd(a, b)$ *is 1.* (Line, 97-98)." We understood, pre-service teacher detected key idea and his detected key idea was 1 of least common multiple.

The first category in the theme of "Metacognitive Skills" was category of "Facilitation the operations". We reached sub-categories of "Change of variable for facilitation the operations", "Avoid fractional expressions for facilitation the operations" and "Detected key idea for proof" in this category. These skills are evaluated as metacognitive in the context of self-regulation, as it is a skill to facilitate it by being aware of the knowledge that one has. Bruning et al. (2014) noted the individual is aware of his or her own skill and explains the regulation of these skills as metacognitive self-regulation. Yüksel (2004) point out the self-regulation of the individual to organize his own skills to achieve his goal. These results confirm earlier findings.

Another category in the theme of metacognitive skills was category of "Questioning". We reached sub-categories of "He/she continues proof steps by asking himself questions", "He/she explains with reason operations", "Questioning for detected purpose", "Questioning whether operation error or not" and "Questioning for controlling correctness of operations".

The first skill in the category of "Questioning" was sub-category of "He/she continues proof steps by asking himself questions". Some of the identified participants exhibiting this skill are presented as an example in Table 10.

Table 10. The statements of some of the identified participants performed the skill of "He/she continues proof steps by asking himself questions"

Participant	Proposition	Expressions
PTL_{M2}	1	"[05.59] … Let n equality to 1, then results is 1. Let n equality to 2, then results is 4. Assume that n equality to (n+1). We must prove n equality to (n+2). Can we show that it's true?" (Line, 44-45)
T_{M2}	2	"[29.28] … I don't get the square root of both sides, but I do the side impact. Can I say the following from here? Can I tell you that $a.b$ divisible c? Yes…" (Line, 176-177).

Table 10 show that the participants continued their process steps by asking themselves questions. In other words, it is understood that the participant exhibits his / her skill of " He/she continues proof steps by asking himself questions."

The second metacognitive skill that is reached in the "Questioning" category was the sub-category of "He/she explains with reason operations." Examples of some of the participants identified as exhibiting this skill are presented in Table 11.

Table 11. The statements of some of the identified participants performed the skill of "He/she explains with reason operations."

Participant	Proposition	Expressions
T_{H1}	1	"[18.16] I can square the number of the terms or say: I can do this because the array is an arithmetic array..." (Line, 118-119).
T_{M2}	2	"[29.28] …$gcd(a, b) = 1$. What does this mean? Greatest common divisor was 1…" (Line 141-142).

Table 11 indicated the participants express their actions by their reasons. In other words, participants explain what they are doing. This situation was interpreted as exhibiting the participants' skill "He/she explains with reason operations."

Another metacognitive skill in this category was sub-category of "Questioning for detected purpose". This skill detected only for Proposition 1. T_{M2} who one of the participants performed this skill said that "[20.16] … *I will show that: Maybe is equality to* $(k + 1)^2$ *total of the number of* $n = (k + 1)$ *odd numbers? Is equality to* $(k + 1)^2$ *total of the number of* $(k + 1)$ *odd numbers?* (Line, 116-117)". This expression show that teacher performed this skill.

Another metacognitive skill in the category of "Questioning" was sub-category of "Questioning whether operation error or not". This skill detected only for Proposition 1. T_{L1} who one of the participants performed this skill said that "[08.11] …$2n^2, 2n, 4n^2$... *Did I make a mistake in there somewhere?* (Line, 41-42)." We interpreted the expression of teacher as skill of questioning whether operation error or not.

The last metacognitive skill in this category was sub-category of "Questioning for controlling correctness of operations". This skill detected only for Proposition 2. T_{H1} who one of the participants performed this skill said, "[18.16] … *Like what from 1 to 55? It is between 1 and* $(k + 1)$. (Line, 119)." The expression of teacher shows that teacher questioning for controlling correctness of operations. We interpreted this case as teachers performed this skill.

We reached sub-categories of "He/she continues proof steps by asking himself questions", "He/she explains with reason operations", "Questioning for detected purpose", "Questioning whether operation error or not" and "Questioning for controlling correctness of operations". This skill assessed as metacognitive, because they based on questioning. Many studies show that these skills is metacognitive skill (Aydın & Ubuz, 2010; Jiang, Ma, & Gao, 2016; Schraw & Dennison, 1994). These results confirm earlier findings.

Another category in the theme of "Metacognitive Skills" was category of "Awareness". This category was detected only for Proposition 2. Sub-categories of "Awareness of proof strategy", "Self-reflection" and "He/she thinking needed proving of all expression in proposition" were detected in this category.

The first metacognitive skill in the category of "Awareness" was sub-category of "Awareness of proof strategy". PTL_{M1} who one of the participants performed this skill said that "[32.41] *I cannot remember the sum of the even numbers, I know that if I can recall the result I will say is the result.* (Line, 81-82)." This expression indicated that pre-service teachers performed this skill. For this reason, it is understood that the pre-service teacher is performed sub-category of "Awareness of proof strategy".

Another metacognitive skill in the category of "Awareness" was sub-category of "Self-reflection". PTL_{H1} who one of the participants performed this skill said that "[25.05] *A proof that mathematical notations are used is a bit like a fake. I have never used the concept of relatively prime numbers. I need to use them. Relatively prime a and b.* (Line, 118-119)." These statements indicate that the person evaluates himself / herself. The sentences of teacher show that teacher performed this skill. Since the self-evaluation of the person in the literature is called reflective thinking, these expressions are interpreted as showing that the preservice teacher exhibits the sub-category of "Self-reflection".

Another metacognitive skill in the category of "Awareness" was sub-category of "He/she thinking needed proving of all expression in proposition". PTL_{H2} who one of the participants performed this skill said that "[30.27] *Normally, I go, but I don't use anything, I do not use relatively prime number, it will probably come out of there. a.m = c, b.n = c, a and b are relatively prime.* (Line, 100-101)." This expression indicated that the teacher is aware that all the data in the proposal can be used for proof.

This category was detected only for Proposition 2. Sub-categories of "Awareness of proof strategy", "Self-reflection" and "He/she thinking needed proving of all expression in proposition" were detected in this category. We assessed these skills as metacognitive in the context of situational knowledge. Many studies show that awareness of when and how will use of a strategy was metacognitive (Mokhtari & Reichard, 2002; Schraw & Dennison, 1994; Yang, 2012). In this context, interpreted as metacognitive of this skill confirm earlier findings.

Another category in the theme of "Metacognitive Skills" was category of "Planning". Sub-categories of "Guessing", "Decides what to prove before proof begins", "Question asking for detected purpose" and "He/she make it in his/her mind first and then roll it" were detected in this category.

The first sub-category in this category was sub-category of "Guessing". It has been seen that PTL_{M2} has decided to target this formula by writing a formula previously known to the participants and then tried to confirm this expression. The activity card display of the participant's "Guessing" skill is presented in Figure 9.

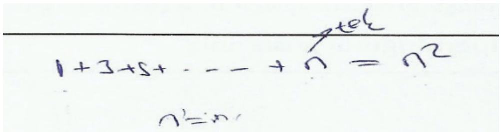

Figure 9. The skill of "Guessing" (PTL_{M2} written the data)

Another metacognitive skill in the category of "Planning" was sub-category of "Decides what to prove before proof begins". T_{M2} who one of the participants performed this skill said that "[24.16] *I think how I should prove it. I try to find the starting point for the proof and determine what I will prove. Now I have to do mathematical proof of what I'm saying. Everything we give in mathematics is supposed to be a surplus...* (Line, 111-113)." This expression indicated that teacher performed this skill.

Another sub-category in the category of "Planning" was skill of "Question asking for detected purpose". T_{H2} who one of the participants performed this skill said that "[24.45] *Now what do we prove? I proved that* $c/(a.b) \in Z$ (Line, 118-119)." This expression indicated that teacher performed skill of "Question asking for detected purpose".

The last metacognitive skill in the category of "Planning" was sub-category of "He/she make it in his/her mind first and then roll it". T_{M2} who one of the participants performed this skill said that:

> "[29.00] Now here. If I pour an expression in my mind on the paper I see the righteousness, and I see the truth, I bring it. When I do operations, I pour my mind design on the paper, and if I see a mistake in one place, this time I try to look at the other side of the mind. (Line, 165-168)".

These sentences show that teacher performed skill of "He/she make it in his/her mind first and then roll it".

Sub-categories of "Guessing", "Decides what to prove before proof begins", "Question asking for detected purpose" and "He/she make it in his/her mind first and then roll it" were detected in this category. Many studies show that these skills is metacognitive skill (Cozza & Oreshkina, 2013; Lesseig, 2016; Zazkis et al., 2016). Interpreted as metacognitive of skills in category of "Planning" confirm earlier findings.

Another category reached in the theme of "Metacognitive skills" was the category of "Strategy determination" category. In this category, we reached the sub-categories of "Divergent thinking ability" and "Convergent thinking ability".The skills of "Divergent thinking ability" and "Convergent thinking ability" were stages of creative thinking. T_{M1} who one of the participants performed skill of creative thinking said that "[18.26] *I'm thinking of a few proof strategies and choosing one of them...* (Line, 93)". This statement indicates that the participant uses the skill of divergent thinking ability and then the convergent thinking ability before determining the strategy.

These two skills, defined, as stages of the creative thinking process, have been as metacognitive evaluated because they require high-level thinking skills. Goldstein (2013) noted creative thinking is more associated with divergent thinking. Divergent thinking allows us to converge to the right thinking and it is confused with convergent thinking. Plotnik (2009, p. 310) and Woolfolk-Hoy (2015, p. 976) emphasized convergent and divergent questions are questions that require analysis, synthesis and evaluation. Schraw and Dennison (1994) point out it is metacognitive skill to choose the best one from these solutions by considering several solutions. These results -evaluated as metacognitive of stage of creative thinking- confirm earlier findings.

Another category in the theme of "Metacognitive skills" was the category of "Controlling". In this category, we detected sub-categories of "Evaluated correctness of proof according to axiomatic proof scheme", "Controlled results purposed whether reach or not", "He/she repeated solved when made wrong operation" and "He/she controlled proof strategy when made mistake".

The first sub-category in the category of "Controlling" was the skill of "Evaluated correctness of proof according to axiomatic proof scheme". Examples of some of the identified participants demonstrating their ability to "Evaluated correctness of proof according to axiomatic proof scheme" are presented in Table 12.

Table 12. The statements of some of the identified participants performed the skill of "Evaluated correctness of proof according to axiomatic proof scheme."

Participant	Proposition	Expressions
T_{M2}	1	"[23.26] Yes, it is true. First of all, I tried to see the formula with simple methods. I tried to show the formula that was formed later by induction method. The Induction method of proof is showing me the truth." (Line, 128-129).
T_{H1}	2	"[24.44] The proof is valid on integers. As a result, a and b are liberal to each other because they are relatively prime. No common divisors. In a number, they are divided into what they say as a multiplier if they pass separately. If they split apart, they show that they have both in c." (Line, 157-160).

Table 12 shows that participants are systematically transmitting within the logical framework of what they are doing. The justification made in this way is called the Axiomatic Proof Scheme, so the skill exhibited by the participants is called "Evaluated correctness of proof according to axiomatic proof scheme." This skill was considered as metacognitive because it is based on the individual's symbolic expression and adequate justification. Dede and Karakuş (2014) noted this proof scheme is a skill that requires deductive reasoning. According to this scheme, the justifying individuals can understand and perceive the new propositions and proofs they meet. For this reason, it can be defined as the top proof scheme when compared to all other proof schemes (Aydoğdu-İskenderoğlu, 2016). In this context, we can say that this result confirms earlier findings.

Another metacognitive skill in this category was the sub-category of "Controlled results purposed whether reach or not". PTL_{H1} who one of the participants performed this skill said that "[14.28] *If any of these made simplification was* $(n-1)^2$. *If* $(n-1)^2 : 4$, *then was* $(n-1) : 2^2$. (Line, 81-82)." Later in the process of think aloud protocol the participant used sentences -"[17.25] *… I got a lot of them. I made it right there. The end would be +1. OK, I guess it turns out right.* (Line, 89-90)."- indicated that pre-service teacher performed skill of "Controlled results purposed whether reach or not". This skill, which is based on checking whether the individual achieves the goal, he has set, has been assessed as metacognitive. Similarity Schraw and Dennison (1994) found controlling whether it has reached its intended outcome or whether the interrogation is metacognitive.

Another metacognitive skill in the category of "Controlling" was the sub-category of "He/she repeated solved when made wrong operation". T_{M2} who one of the participants performed this skill said that "[33.55] *I'm trying to get to the end right now. I do not see a place, I realize this. I turn to the question again.* (Line, 187-188)". This expression indicated that teacher if made wrong operation, she repeated solving the problem. This skill is need monitoring of solving process. Thus, it interpreted as metacognitive. Similarity Nool (2012) reported pre-service mathematics teacher are solving the problem by returning to the beginning when the problem cannot be solved in the non-routine problem solving process.

The last metacognitive skill in this category was the sub-category of "He/she controlled proof strategy when made mistake". T_{H2} who one of the participants performed this skill said that "[31.56] *At first I made a mistake on this road, I am trying to control it. If there is an error, I will try another strategy*. (Line, 132-133)" The expression of the teacher show that she performed this skill. This skill has needed controlling and monitoring of proving process. Therefore, it considered as metacognitive. Similarity Fang and Cox (1999) demonstrated controlling after determining of proof strategy was metacognitive.

Another category in the theme of "Metacognitive skills" was the category of "Relationship". In this category we detected that participants performed sub-categories of "Relationship between proof steps" and "Relationship while question asking".

The first metacognitive skill in the category of "Relationship" was sub-category of "Relationship between proof steps". Examples of some of the identified participants demonstrating their ability to "Relationship between proof steps" are presented in Table 13.

Table 13. The statements of some of the identified participants performed the skill of "Relationship between proof steps."

Participant	Proposition	Expressions
T_{H1}	1	"[14.47] … The other side of equality has made $2a$. Now $2k + 2$ is related to the number of terms. I have to go to the number of terms... (Line, 105-106)."
PTL_{H2}	2	"[28.42] The expression I write does not seem to mean much to prove. The expression seemed to help. But what I want to find is that c is completely divisible to m? I want to find out exactly where c is divided into m above. I will have completed the test if I found him, so I did something like that." (Lines, 92-95)

When the Table 13 are examined, it is understood that they demonstrate the skill of "Relationship between proof steps". In other words, the participants are trying to establish a connection between the process they have done in one step and the process they have done in the next step. This situation makes the participants "Relationship between proof steps.". It is interpreted as exhibiting his skill.

Other metacognitive skill in this category sub-category of "Relationship while question asking". T_{M2} was of the participants performed the skill of "Relationship while question asking" in the Proposition 1. He said that "[20.16] *Now here is what I have to pay attention to here: one k-number missing in* $k^2 + k + 1$. *Where do I get this k-number?* (Line, 119-121)." T_{M2} was of the participants performed this skill in the Proposition 2. For example she said that "[26.15] *…The last I want to show is* $a.k = b.l$. *I know something more that it was* $lcm(a, b) = 1$. (Line, 154-155). *What can I say about* k *and* l*?* (Line, 156)". This expression indicated that participants performed this skill.

We assessed these skills as metacognitive because they require high-level reasoning. Many studies show that these skills were metacognitive (Cozza & Oreshkina, 2013; Yang, 2012; Zazkis et al., 2016). Interpreted as metacognitive of skills in category of "Relationship" confirm earlier findings.

The last category in the theme of "Metacognitive skills" was the category of "Analogical reasoning". In this category, we detected sub-categories of "He makes analogical reasoning by changing the strategy he has used before", "Made analogical reasoning benefit from complement of a set" and "Made analogical reasoning benefit from contradiction of proposition".

The first metacognitive skill in this category was the sub-category of "He makes analogical reasoning by changing the strategy he has used before". Examples of some of the identified participants demonstrating their ability to "He makes analogical reasoning by changing the strategy he has used before" are presented in Table 14.

Table 14. The statements of some of the identified participants performed the skill of "He makes analogical reasoning by changing the strategy he has used before"

Participant	Proposition	Expressions
T_{H1}	1	"[14.47] …I also wrote array opposite to it. I have not actually proved this before. I will use the logic I use from the sum of numbers from 1 to n... (Line, 101-102)."
PTL_{M1}	2	"[32.50] I have tried to do this when you ask the previous question, but again I have given something like 2k, 2k + 2, so I recall. Therefore, I adapt it to that." (Line, 84-86).

Table 14 show that the participants changed to the solution that they had used and reached the right result and adapted new probing. This situation was interpreted as exhibiting the skill of the participants "He makes analogical reasoning by changing the strategy he has used before."

Another sub-category in this category was the skill of "Made analogical reasoning benefit from complement of a set". This sub-category was performed only for Proposition 1. For example, PT1$_{L2}$ said that "[30.50] *Yes, I work on the thing, if I recall the even number of this term, if I recall the consecutive even numbers, I said I would take the odd numbers out, but I'm not sure of the double numbers either.* (Line, 216-218)". This expression indicated that participants performed this skill.

The last metacognitive skill in this category was the sub-category of "Made analogical reasoning benefit from contradiction of proposition". T$_{H2}$ was one of the participants performed this skill. Her expression as following:

> "[25.17] Such a proposition, yes, actually, I say: We use more inversions, for example, what we are saying is that if a number is divisible by 12, it can be divided by 3 and 4. This is the high school we used more often. If it is divided by 24, it is divided by 8 and 3. In fact, this is what we use more. But that is to say, reasonably also lies in the mind; If it is divided in pieces, it has to be divided in the whole (Line, 162-166)"

When the statements of the teacher are examined, he states that he had met the opposite with the inverse of proposition. In other words, it is understood that the teacher knows the proof in contrast to this double-sided proposition. When the teacher' statements are examined, it is understood that she performed the skill of "Made analogical reasoning benefit from contradiction of proposition".

In this category, we detected sub-categories of "He makes analogical reasoning by changing the strategy he has used before", "Made analogical reasoning benefit from complement of a set" and "Made analogical reasoning benefit from contradiction of proposition". These skills require the ability to reason and associate with a method that one has already used because it requires adaptation of the new problem. Thus, we interpreted it as metacognitive. Similarity, many studies show that analogical reasoning is metacognitive skill (Goldstein, 2013; Smith & Kosslyn, 2014).

Table 15 indicated that distribution in according to propositions and participants of the sub-category of theme of "Metacognitive skills".

Table 15. Distribution in according to propositions and participants of the sub-category of theme of "Metacognitive skills".

Category	Sub-category	Propositions		Performed skill teachers		Performed skill last grade pre-service teachers		Performed skill 1st grade pre-service teachers	
		f	No	f	Teachers	f	Pre-service Teachers	f	Pre-service Teachers
Facilitation the operations	Change of variable for facilitation the operations	2	1.S	1	Ö$_{i1}$	1	ÖAS$_{i2}$	-	-
			2.S	1	Ö$_{i2}$	2	ÖAS$_{D2}$- ÖAS$_{O1}$	-	-
	Avoid fractional expressions for facilitation the operations	1	2.S	-	-	1	ÖAS$_{D1}$	-	-
	Detected key idea for proof	1	2.S	1	Ö$_{i1}$	1	ÖAS$_{i1}$	-	-
Questioning	He/she continues proof steps by asking himself questions	2	1.S	2	Ö$_{D1}$- Ö$_{i1}$	3	ÖAS$_{D1}$- ÖAS$_{O2}$- ÖAS$_{i1}$	-	-
			2.S	1	Ö$_{O2}$	2	ÖAS$_{D1}$-ÖAS$_{O1}$	-	-
	He/she explains with reason operations	2	1.S	1	Ö$_{i1}$	-	-	-	-
			2.S	1	Ö$_{O2}$	1	ÖAS$_{D1}$	-	-
	Questioning for detected purpose	1	1.S	1	Ö$_{O2}$	-	-	-	-
	Questioning whether operation error or not	1	1.S	1	Ö$_{D1}$	-	-	-	-
	Questioning for controlling correctness of operations	1	2.S	1	Ö$_{i1}$	-	-	-	-
Awareness	Awareness of proof strategy	1	2.S	-	-	1	ÖAS$_{O1}$	-	-
	Self-reflection	1	2.S	1	Ö$_{O1}$	1	ÖAS$_{i1}$	-	-
	He/she thinking needed proving of all expression in proposition	1	2.S	-	-	1	ÖAS$_{i2}$	-	-
Planning	Guessing	1	1.S	-	-	1	ÖAS$_{O2}$	-	-
	Decides what to prove before proof begins	1	2.S	1	Ö$_{O1}$	-	-	-	-
	Question asking for detected purpose	1	2.S	2	Ö$_{O2}$- Ö$_{i2}$	-	-	-	-
	He/she make it in his/her mind first and then roll it	1	2.S	1	Ö$_{O2}$	-	-	-	-
Strategy determination	Divergent thinking ability	1	1.S	1	Ö$_{O1}$	1	ÖAS$_{i1}$	-	-
	Convergent thinking ability	1	1.S	1	Ö$_{O1}$	1	ÖAS$_{i1}$	-	-
Controlling	Evaluated correctness of proof according to axiomatic proof scheme	2	1.S	2	Ö$_{O2}$- Ö$_{i2}$	3	ÖAS$_{O1}$- ÖAS$_{i1}$- ÖAS$_{i2}$	-	-
			2.S	1	Ö$_{i1}$	-	-	-	-
	Controlled results purposed whether reach or not	1	1.S	-	-	1	ÖAS$_{i1}$	-	-
	He/she repeated solved when made wrong operation	1	2.S	1	Ö$_{O2}$	-	-	-	-
	He/she controlled proof strategy when made mistake	1	2.S	1	Ö$_{i2}$	-	-	-	-

Table 15. Continued

Category	Sub-category	Propositions		Performed skill teachers		Performed skill last grade pre-service teachers		Performed skill 1st grade pre-service teachers	
		f	**No**	**f**	**Teachers**	**f**	**Pre-service Teachers**	**f**	**Pre-service Teachers**
Relationship	Relationship between proof steps	2	1.S	1	Ö$_{i1}$	-	-	-	-
			2.S	2	Ö$_{O1}$- Ö$_{i1}$	1	ÖAS$_{i2}$	-	-
	Relationship while question asking	2	1.S	1	Ö$_{O2}$	1	ÖAS$_{i1}$	-	-
			2.S	1	Ö$_{O2}$	-	ÖAS$_{i1}$	-	-
Analogical reasoning	He makes analogical reasoning by changing the strategy he has used before	2	1.S	2	Ö$_{i1}$- Ö$_{i2}$	2	ÖAS$_{O2}$- ÖAS$_{i1}$	-	-
			2.S	-	-	2	ÖAS$_{O1}$	1	ÖA1$_{i2}$
	Made analogical reasoning benefit from complement of a set	1	1.S	1	Ö$_{D1}$	-	-	1	ÖA1$_{i2}$
	Made analogical reasoning benefit from contradiction of proposition	1	2.S	1	Ö$_{i1}$	-	-	-	-

Table 15 show that the most performed skill was "He/she continues proof steps by asking himself questions" in theme of metacognitive skills. The least performed skills were "Avoid fractional expressions for facilitation the operations", "Questioning for detected purpose", "Questioning whether operation error or not", "Questioning for controlling correctness of operations", "Awareness of proof strategy", "He/she thinking needed proving of all expression in proposition", "Guessing", "Decides what to prove before proof begins", "He/she make it in his/her mind first and then roll it", "Controlled results purposed whether reach or not", "He/she repeated solved when made wrong operation", "He/she controlled proof strategy when made mistake", "Made analogical reasoning benefit from contradiction of proposition". The category which is determined to be the most frequently exhibited category in terms of categories is the category of "Questioning", while the least exhibited category is the category "Strategy determination".

Results and Discussion on Related to Qualitative and Quantitative Data

The quantitative findings show that points of proving test of teachers and last grade preservice teachers significantly higher than points of proving test of first grade pre-service teachers. The number of teachers who performed similar metacognitive skill prove the diagnostic test. Therefore, teachers and last grade preservice teachers show greater metacognitive skills than first grade preservice teachers. Similar results were found in the teachers and preservice teachers' cognitive skills. However, a greater difference exists between the groups' metacognitive skills than their cognitive skills. Özsoy and Günindi (2011) noted it differs significantly in favor of the 4th grade pre-service teacher metacognitive awareness of pre-service pre-school teachers. Tüysüz, Karakuyu, and Bilgin (2008) point out the level of metacognition of pre-service classroom teacher was increasing due to the increase in class levels. The findings of this study that the "Proving Diagnostic Test" scores and the cognitive-metacognitive skills scores of the 1st grade pre-service teachers are the least and the results of the "Proving Diagnostic Test" scores of the teachers and the cognitive-metacognitive skills scores are supported by these studies. As seen in qualitative data, skills of participants in proving process created two theme as cognitive skills and metacognitive skills.

Conclusion and Suggestions

The present study examined performed skill in proving process of pre-service and service teachers conducted two sections that they are quantitative and qualitative. Results of quantitative data indicated that proving skill of teachers and the last grade pre-service math teacher more than first grade pre-service teachers. The numbers of cognitive and metacognitive skills in proving process similarity point of "Proving Diagnostic Test". The number of cognitive and metacognitive skills exhibited in the proof-making process was found to be similar to the scores obtained from the proving diagnostic test. In other words, while the number of cognitive and metacognitive skills performed during the proof-making period is the highest in teachers, and then in the last grade of teacher candidates, the least number of skills displayed is in the 1st grade teacher candidates.

As seen in the cognitive skills theme, this theme is gathered in five categories as "Read the proposition of the proof", "Evaluating the correctness", "Determining strategy", "Carry out plans" and "Heuristic shortcuts". As seen in the metacognitive skills theme, this theme is gathered in eight categories as "Facilitation the operations", "Questioning", "Awareness", "Planning", "Strategy determination", "Controlling", "Relationship" and "Analogical reasoning".

If the quantitative results of the study are assessed as developmental, we can say that as the education level of the individual increases, the ability to prove also increases. Moreover, the fact that teachers' scores of proving skills are higher than preservice teachers shows that their ability to prove themselves has improved when they continue to their professional lives. The same is true for the number of cognitive and metacognitive skills that they exhibit during the proving process. We interpreted this position as the fact that the preservice teacher at the undergraduate level have formal proof while the teachers have made informal proof. Arcavi et al. (2017) stated that teachers have two different perspectives, one of which must be their own perspective, and the other must be a perspective of students. Because teachers are constantly in the practice environment, it is also possible to look for proof according to the point of view of the students. This may have made them aware of what they are doing by taking it out of formalism, revealing meaning and exhibiting cognitive skills.

Findings show that mathematics teachers and prospective teachers use cognitive skills more in the process of proving. Metacognitive skills are less used. This finding indicates that participants did not know what they were doing when they were doing the proof and did not make enough inquiries and evaluations. This case can be interpreted as the fact that pre-service math teachers do not learn enough to prove the course in proof teaching. In order to solve this problem, it is thought that the first step is to destroy the perception that is a concept that can be done by memorizing the proof in the individual. For this reason, teachers must give up teaching in their class and teach the pupil's reasoning, and play a guiding role in which their own proofs can organize themselves. Instructors conducting the teaching of proofs should provide awareness of the learner by asking questions about the proof process, explaining why they did it during an operation, and making an evaluation after completing the procedure. Teachers can also be given in-service trainings and awareness about how to teach them. Because metacognitive skills can be taught, and learning of these skills helps someone to understand spontaneity, accelerates learning.

Another result show that participants should not resort to intuitive shortcuts because they do not know how to use in the lemma should be used stage. This case indicated curriculum must cover but only course that made proof, not also course that made teaching proof. In the present curriculum, teaching proof has made Abstract Mathematics course, but time is not enough to it. For this reason, a lesson on proof teaching should be put on the abstract mathematics course and necessary activities should be developed. The participants had difficulty using mathematical notations, and very few had justified it according to the axiomatic proof scheme. In addition, this case show that inclusion curriculum of a course with teaching proof is necessary.

This study show that related to achievement of proof and the numbers of performed metacognitive skill. However, we out of scope of study question that the development of metacognitive skills increases the success of proof, or the increase in the success of proof increases the number of metacognitive skills. Future researchers can search for solutions to these problems by conducting causal comparison studies and empirical studies.

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